Lecture 13 18-10-18

Alternative way: Compute $\left(e^{-\alpha t} f\left(\frac{d}{dt}\right) e^{\alpha t}\right)$ as an operator.

write: $f\left(\frac{d}{dt}\right) = a_n\left(\frac{d}{dt} - \lambda_1\right) \cdots \cdots \left(\frac{d}{dt} - \lambda_n\right)$

for some $\lambda_i \in \mathbb{C}$, not necessary distinct

$$e^{-\alpha t}\left(\frac{d}{dt} - \lambda_i\right) e^{\alpha t} = e^{-\alpha t}\left(e^{\lambda_i t} \frac{d}{dt} e^{-\lambda_i t}\right) e^{\alpha t}$$

$$= \left(\frac{d}{dt} + (\alpha - \lambda_i)\right)$$

Now: $e^{-\alpha t} f\left(\frac{d}{dt}\right) e^{\alpha t}$

$$= e^{-\alpha t}\left[a_n\left(\frac{d}{dt} - \lambda_1\right) \cdots \left(\frac{d}{dt} - \lambda_n\right)\right] e^{\alpha t}$$

$$= a_n\left(e^{-\alpha t}\left(\frac{d}{dt} - \lambda_1\right) e^{\alpha t}\right)\left(e^{-\alpha t}\left(\frac{d}{dt} - \lambda_2\right) e^{\alpha t}\right) \cdots$$

$$\cdots \left(e^{-\alpha t}\left(\frac{d}{dt} - \lambda_n\right) e^{\alpha t}\right)$$

$$= a_n\left(\frac{d}{dt} + \alpha - \lambda_1\right) \cdots \cdots \left(\frac{d}{dt} + \alpha - \lambda_n\right)$$

- If we let $g(r) = f(r + \alpha)$.

we know $e^{-\alpha t} f\left(\frac{d}{dt}\right) e^{\alpha t} = g\left(\frac{d}{dt}\right)$

Come back to the original problem:

Want: $$f\left(\frac{d}{dt}\right)\left(e^{\alpha t}Q_l(t)\right) = e^{\alpha t}P_k(t)$$

$$\Longleftrightarrow \quad \left(e^{-\alpha t} f\left(\frac{d}{dt}\right) e^{\alpha t}\right)\left(Q_l(t)\right) = P_k(t).$$

$$\Longleftrightarrow \quad g\left(\frac{d}{dt}\right)\left(Q_l(t)\right) = P_k(t).$$

with: $$g(r) = \frac{f^{(n)}(\alpha)}{n!} r^n + \dots + f'(\alpha) r + f(\alpha)$$

☆ Back to the previous case!

If we take $n=2$: $$f(r) = ar^2 + br + c$$

$$g(r) = ar^2 + (2a\alpha + b) r + (a\alpha^2 + b\alpha + c)$$
$$= ar^2 + f'(\alpha) r + f(\alpha).$$

Equation to solve:

$$a(j+2)(j+1)B_{j+2} + (j+1) f'(\alpha) B_{j+1} + f(\alpha) B_j = A_j$$

Which is the equation we get when we talk about 2^{nd} order equation.

Finally: $r(t) = e^{\alpha t}\cos\mu t \, P_k(t)$, or $e^{\alpha t}\sin\mu t \, P_k(t)$

$$e^{\alpha t}\cos\mu t = \left(\frac{e^{\lambda t}+e^{\bar{\lambda} t}}{2}\right), \quad e^{\alpha t}\sin\mu t = \left(\frac{e^{\lambda t}-e^{\bar{\lambda} t}}{2i}\right)$$

then we only have to consider

$$r(t) = e^{\lambda t} P_k(t) \quad \text{with} \quad \lambda = \alpha + i\mu.$$

Letting: $g(r) = f(r+\lambda)$ and we solve for

$$g\left(\frac{d}{dt}\right)(Q_\ell(t)) = P_k(t).$$

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§ Variational of parameter:

Equation: $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_0(t)\, y = r(t)$ ----- $(**)$

Assume: $y_1, \dots, y_n$ fundamental set of solutions to to homogeneous equation

Q: How to find a particular solution to $(**)$?

Idea: Try $Y(t) = u_1(t)\, y_1(t) + \cdots + u_n(t)\, y_n(t)$.
and plug into $(**)$.

$$Y'(t) = u_1' y_1 + \cdots + u_n' y_n + u_1 y_1' + \cdots + u_n y_n'$$

Like in the case for 2^{nd} order equation, we let

$$u_1' y_1 + \cdots + u_n' y_n = 0$$

$$\Rightarrow \quad Y'(t) = u_1 y_1' + \cdots + u_n y_n'$$

$$Y''(t) = u_1' y_1' + \cdots + u_n' y_n' + u_1 y_1'' + \cdots + u_n y_n''$$

<u>Set:</u> $\quad u_1' y_1' + \cdots + u_n' y_n' = 0$

$\vdots$

$$Y^{(j)} = u_1 y_1^{(j)} + \cdots u_n y_n^{(j)}$$

$$Y^{(j+1)} = u_1' y_1^{(j)} + \cdots + u_n' y_n^{(j)} + u_1 y_1^{(j+1)} + \cdots + u_n y_n^{(j+1)}$$

<u>Set:</u> $\quad u_1' y_1^{(j)} + \cdots + u_n' y_n^{(j)} = 0.$

$\vdots$

$$Y^{(n-1)} = u_1 y^{(n-1)} + \cdots + u_n y^{(n-1)}$$

$$Y^{(n)} = u_1' y^{(n-1)} + \cdots + u_n' y^{(n-1)} + u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}$$

Plug in **: $\quad Y^{(n)}(t) + p_{n-1}(t)\, Y^{(n-1)} + \dots + p_0(t)\, Y = r(t)$

$$
\begin{aligned}
u_1' y^{(n-1)} + \dots + u_n' y^{(n-1)} &+ u_1 \left(\cancelto{0}{y_1^{(n)} + p_{n-1} y_1^{(n-1)} + \dots + p_0(t) y_1}\right) \\
&+ u_2 \left(\cancelto{0}{y_2^{(n)} + p_{n-1} y_2^{(n-1)} + \dots + p_0(t) y_2}\right) \\
&\ \ \vdots \\
&+ u_n \left(\cancelto{0}{y_n^{(n)} + p_{n-1} y_n^{(n-1)} + \dots + p_0(t) y_n}\right) \\
&= r(t).
\end{aligned}
$$

$\Rightarrow$ We get system of equation:

$$
\underbrace{\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \ddots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}}_{= M(t).}
\begin{pmatrix} u_1' \\ \vdots \\ \vdots \\ u_n' \end{pmatrix}
=
\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ r \end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}' = M(t)^{-1} \begin{pmatrix} 0 \\ \vdots \\ r \end{pmatrix} = \begin{pmatrix} I_1(t)\, r(t) \\ \vdots \\ I_n(t)\, r(t) \end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix} | \\ u \\ | \end{pmatrix} = \int \begin{pmatrix} I_1(t)\, r(t) \\ \vdots \\ I_n(t)\, r(t) \end{pmatrix} dt + \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}
$$

Q: What is $I_1(t), \dots, I_n(t)$?

$$
I_i(t) = \frac{1}{\det M(t)} \det \begin{pmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & 1 & \cdots & y_n^{(n-1)} \end{pmatrix}
$$

i-th column

Formula:

$$Y(t) = y_1 \int \frac{I_1(t)\, r(t)}{W(t)}\, dt + \cdots + y_n \int \frac{I_n(t)\, r(t)}{W(t)}\, dt$$

Rk: The ordering $y_1, \ldots, y_n$ in the above formula Need to argue with the ordering defining $M(t)$!

Eg. $y^{(3)} + y^{(1)} = \frac{1}{\cos^2(t)}$ on $I = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Step 1: Solve homogeneous eqt,

$$y_1 = 1,\ y_2 = \cos t,\ y_3 = \sin(t).$$

Step 2: Find

$$M(t) = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix}$$

$$W(t) = 1$$

$$I_1(t) = \det\begin{pmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{pmatrix} = 1$$

$$I_2(t) = \det\begin{pmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{pmatrix} = -\cos t$$

$$I_3(t) = \det\begin{pmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{pmatrix} = -\sin t.$$

$$\Rightarrow \quad u_1 = \int \frac{1}{\cos^2 t}\,dt = \tan t$$

$$u_2 = \int \frac{-\cos t}{\cos^2 t}\,dt = -\log|\sec t + \tan t|$$

$$u_3 = \int \frac{-\sin t}{\cos^2 t}\,dt = -\sec(t)$$

<u>Plug in:</u> $Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3$

and get the solution!